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Orbital Decay Due to Drag in an Exponentially Varying Atmosphere

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Introduction

In order to illustrate the use of multiple variable expansions in satellite problems, Kevorkian[1] studied the idealized problem of the orbital decay of a satellite with constant aerodynamic coefficients in a thin constant-density atmosphere. The assumption of a constant density, which is admittedly physically unrealistic, leads to considerable simplification. It implies that the order of magnitude of the aerodynamic forces remains unchanged as the orbit evolves. Consequently, one can account for the small but cumulative aerodynamic perturbations for all times by means of one uniformly valid multivariable expansion. In the more realistic problem modeled by an exponentially varying density, this is no longer the case. A glide trajectory which originates at a sufficiently high altitude will remain only for a limited time in a regime (outer region) where the aerodynamic forces are small in comparison with the gravitational attraction. Eventually, as the orbit decays due to drag, the relative orders of magnitude of the aerodynamic vs gravitational terms will reverse (inner region). This type of problem requires the matching of an outer and inner expansion. Examples are studied in Refs. 2-4.

If the initial conditions are such that the glider completes a large number of orbits before entry into the inner region, one must still use a multiple variable expansion (or an equivalent formulation) to describe the outer solution uniformly, even though its duration is limited.

In this Note, we consider the simple problem of extending Kevorkian's results to an exponentially varying atmosphere in the outer region. We do not consider the solution in the inner region.

Mathematical Model

A planar motion is considered using the polar coordinates (R,θ) as follows[1]:

$$m\frac{\mathrm{d}^2R}{\mathrm{d}T^2}-mR\left(\frac{\mathrm{d}\theta}{\mathrm{d}T}\right)^2=-\frac{GmM}{R^2}-D\sin\gamma \tag{1}$$

$$mR\frac{\mathrm{d}^2\theta}{\mathrm{d}T^2}+2m\frac{\mathrm{d}R}{\mathrm{d}T}\frac{\mathrm{d}\theta}{\mathrm{d}T}=-D\cos\gamma \tag{2}$$

where m is the mass of the satellite, M the mass of the Earth, D the drag force, and γ the flight-path angle.

Now, D is given by

$$D=\tfrac{1}{2}\rho V^2SC_D \tag{3}$$

where ρ is the atmospheric density, S the cross-sectional area of the satellite, and C_D the constant drag coefficient. The magnitude of the velocity is

$$V=[(\mathrm{d}R/\mathrm{d}T)^2+R^2(\mathrm{d}\theta/\mathrm{d}T)^2]^{1/2} \tag{4}$$

and the flight-path angle γ is

$$\gamma=\tan^{-1}\frac{\mathrm{d}R/\mathrm{d}T}{R\mathrm{d}\theta/\mathrm{d}T} \tag{5}$$

The exponential density model can be written as

$$\rho=\rho_0e^{-(R-R_0)/H}\qquad \rho_0=\rho(R_0) \tag{6}$$

where R_0 is a reference radius and H the scale height of the atmosphere. Fitting Eq. (6) to any two values of ρ at two different altitudes fixes ρ_0 and H. In terms of the nondimensional parameters $r=R/R_0$, θ, and $t=T/(R_0^3/GM)^{1/2}$, the equations of motion become

$$\ddot{r}-r\dot{\theta}^2=-1/r^2-\epsilon e^{-(r-1)/H_s}\dot{r}(\dot{r}^2+r^2\dot{\theta}^2)^{1/2} \tag{7}$$

$$r\ddot{\theta}+2\dot{r}\dot{\theta}=-\epsilon e^{-(r-1)/H_s}r\dot{\theta}(\dot{r}^2+r^2\dot{\theta}^2)^{1/2} \tag{8}$$

where $H_S=H/R_0$, and $\epsilon=C_D\rho_0SR_0/2m$ and is small if ρ_0 is small. Transforming Eqs. (7) and (8) to $u(\theta)$ and $t(\theta)$ with $u=1/r$ then gives

$$u''+u-u^4t'^2=0 \tag{9}$$

$$(u^2t')'=\epsilon e^{-(1-u)/H_su}t'(u'^2+u^2)^{1/2} \tag{10}$$

where u is a harmonic function of θ if $\epsilon=0^1$, and $(\ )'=\mathrm{d}/\mathrm{d}\theta$.

The initial conditions we adopt correspond to the satellite being at pericenter at $t=0$. Also, with no loss of generality, we may choose the argument of pericenter $\omega=0$. Hence, the initial $(\theta=0)$ conditions are

$$u(0)=1\qquad t(0)=0\qquad u'(0)=0\qquad t'(0)=\sigma<1 \tag{11}$$

where σ is the reciprocal angular velocity initially.

If $\epsilon=0$, the above initial conditions define a unique Keplerian ellipse with constant elements a, e, ω, and τ, where a, e, and τ are the semimajor axis, the eccentricity, and the time of passage through pericenter, respectively. With $\epsilon\neq0$ and to order unity, that is, considering the first term in a series expansion in powers of ϵ, the motion will still be in the form of a Keplerian orbit but with slowly varying elements. It is, therefore, convenient to express the initial conditions of Eq. (11) in terms of equivalent conditions on the initial values of a, e, ω, and τ.

As the satellite starts at the pericenter, $\tau(0)=0$ from Kepler's equation for the time history of the orbit. Moreover, we chose $\omega(0)=0$. Since the pericenter distance is $a(1-e)$, we have

$$a(0)[1-e(0)]=1 \tag{12}$$

Differentiating Kepler's equation for the time history with respect to θ and using the conventional definition for the eccentric anomaly gives

$$\sigma=1/\sqrt{1+e(0)} \tag{13}$$

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Thus, the initial values of the slowly varying elements are

$$a(0)=\sigma^2/(2\sigma^2-1) \qquad e(0)=(1-\sigma^2)/\sigma^2$$

$$\tau(0)=0 \qquad \omega(0)=0 \tag{14}$$

Two-Variable Solution

We develop u and t' as follows:

$$u(\theta,\epsilon)=u_0(\theta,\tilde{\theta})+\epsilon u_1(\theta,\tilde{\theta})+\cdots \tag{15a}$$

$$t'(\theta,\epsilon)=v_0(\theta,\tilde{\theta})+\epsilon v_1(\theta,\tilde{\theta})+\cdots \tag{15b}$$

where $\tilde{\theta}=\epsilon\theta$ is the slow variable.

Substituting Eqs. (15) and the corresponding formulas for the derivatives into Eqs. (9) and (10) gives

$$\partial^2 u_0/\partial\theta^2+u_0=u_0^4 v_0^2 \tag{16}$$

$$u_0^2\frac{\partial v_0}{\partial\theta}+2u_0\frac{\partial u_0}{\partial\theta}v_0=0 \tag{17}$$

$$\frac{\partial^2 u_1}{\partial\theta^2}+u_1=-2\frac{\partial^2 u_0}{\partial\theta\partial\tilde{\theta}}+2u_0^4 v_0 v_1+4u_0^3 u_1 v_0^2 \tag{18}$$

$$u_0^2\left(\frac{\partial v_1}{\partial\theta}+\frac{\partial v_0}{\partial\tilde{\theta}}\right)+2u_0u_1\frac{\partial v_0}{\partial\theta}+2u_1\frac{\partial u_0}{\partial\theta}v_0+2u_0\left[\frac{\partial u_0}{\partial\theta}v_1+v_0\left(\frac{\partial u_1}{\partial\theta}+\frac{\partial u_0}{\partial\tilde{\theta}}\right)\right]=e^{-(1-u_0)/H_s u_0}v_0\left[\left(\frac{\partial u_0}{\partial\theta}\right)^2+u_0^2\right]^{1/2} \tag{19}$$

The solution of Eqs. (16) and (17) is most conveniently expressed in terms of the three slowly varying parameters $a(\tilde{\theta})$, $e(\tilde{\theta})$, and $\omega(\tilde{\theta})$ as follows:

$$u_0(\theta,\tilde{\theta})=p^2[1+e\cos(\theta-\omega)] \tag{20a}$$

$$v_0(\theta,\tilde{\theta})=p^{-3}[1+e\cos(\theta-\omega)]^{-2} \tag{20b}$$

where p is the reciprocal angular momentum, $p=1/\sqrt{a(1-e^2)}$. We shall henceforth assume $e(0)$ to be small and neglect terms of the order $e^2(0)$. It should be borne in mind that e is a monotonically decreasing function of $\tilde{\theta}$.

Equation (19) can be integrated once with respect to θ if we make use of Eqs. 20. We find

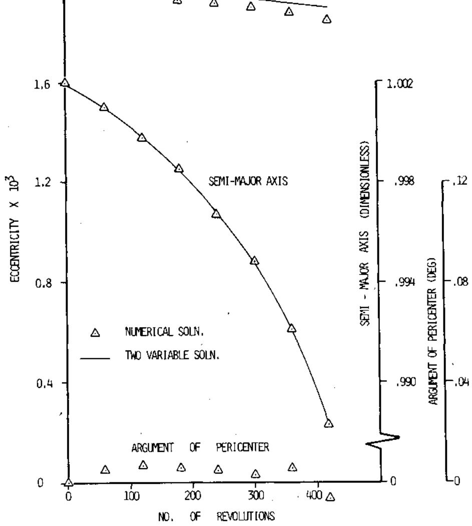

Fig. 1 Slowly varying elements.

$$u_0^2 v_1+2p\frac{u_1}{u_0}+\frac{\mathrm{d}p}{\mathrm{d}\tilde{\theta}}\theta-\frac{1}{p}\exp(\beta)\left[\theta-e\sin(\theta-\omega)\right]$$

$$=g_1(\tilde{\theta})+O(e^2) \tag{21}$$

where g_1 is an unknown function of $\tilde{\theta}$ and $\beta=(p^2-1)/H_s p^2$. Using Eqs. (21) in Eq. (18) leads to

$$\frac{\partial^2 u_1}{\partial\theta^2}+u_1=\left[4p\frac{\mathrm{d}p}{\mathrm{d}\tilde{\theta}}e+2p^2\frac{\mathrm{d}e}{\mathrm{d}\tilde{\theta}}-2e\exp(\beta)\right]\sin(\theta-\omega)$$

$$-2p^2e\frac{\mathrm{d}\omega}{\mathrm{d}\tilde{\theta}}\cos(\theta-\omega)+2\left[\exp(\beta)-p\frac{\mathrm{d}p}{\mathrm{d}\tilde{\theta}}\right]\theta+2pg_1(\tilde{\theta})+O(e^2) \tag{22}$$

Now unless u_1 is bounded, the assumed expansion for u would be inconsistent. Therefore, we must set

$$\exp(\beta)-p\mathrm{d}p/\mathrm{d}\tilde{\theta}=0 \tag{23}$$

$$2pe\mathrm{d}p/\mathrm{d}\tilde{\theta}+p^2\mathrm{d}e/\mathrm{d}\tilde{\theta}-e\exp(\beta)=0 \tag{24}$$

$$p^2e\mathrm{d}\omega/\mathrm{d}\tilde{\theta}=0 \tag{25}$$

On solving these and remembering that $e(0)$ and $|1-1/u_0|$ are small as compared to 1, we find

$$p(\tilde{\theta})=\left\{1+H_s\ln\left[\exp\frac{e(0)}{H_s}-\frac{2\tilde{\theta}}{H_s}\right]\right\}^{-1/2} \tag{26}$$

$$e(\tilde{\theta})=\frac{e(0)}{\sqrt{1+e(0)}}\left\{1+H_s\ln\left[\exp\frac{e(0)}{H_s}-\frac{2\tilde{\theta}}{H_s}\right]\right\}^{1/2} \tag{27}$$

$$a(\tilde{\theta})=\left\{1+H_s\ln\left[\exp\frac{e(0)}{H_s}-\frac{2\tilde{\theta}}{H_s}\right]\right\} \tag{28}$$

$$\omega(\tilde{\theta})=0 \tag{29}$$

This shows that, to order unity, the effect of drag leaves the pericenter fixed but produces a decay in the orbit and a decrease of the eccentricity.

Numerical Example and Results

We select $e(0)=0.002$, initial altitude $=500{,}000$ ft and the density model valid between 500,000 and 2,000,000 ft. Also, we select $C_D=0.04$, $S=3.985$ ft^2 and $m=3.8$ slugs. It then follows that $H=181{,}372$ ft, $R_0=21{,}478{,}100$ ft, $H_s=0.008$, $\epsilon=1.4\times10^{-6}$, $\sigma=0.999$ and $a(0)=1.002$. Finally, $a(\tilde{\theta})$, $e(\tilde{\theta})$ and $\omega(\tilde{\theta})$, as obtained from Eqs. (27-29) are plotted and compared with the corresponding values obtained by the numerical integration of Eqs. (9) and (10) by a fourth-order variable step Runge-Kutta technique and use of Eq. (20a), its derivative, and Eq. (20b). As shown in Fig. 1, the two sets of values are found to agree quite well.

Acknowledgments

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